

Inverse Conductivity Problem for a Parabolic Equation using a Carleman Estimate with One Observation

February 1, 2008

PATRICIA GAITAN

Laboratoire d'Analyse, Topologie, Probabilités
CNRS UMR 6632, Marseille, France and Université de la Méditerranée

Abstract

For the heat equation in a bounded domain we give a stability result for a smooth diffusion coefficient. The key ingredients are a global Carleman-type estimate, a Poincaré-type estimate and an energy estimate with a single observation acting on a part of the boundary.

1 Introduction

This paper is devoted to the identification of the diffusion coefficient in the heat equation using the least number of observations as possible.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of \mathbb{R}^n with $n \leq 3$, (the assumption $n \leq 3$ is necessary in order to obtain the appropriate regularity for the solution using classical Sobolev embedding, see Brezis [3]). We denote $\Gamma = \partial\Omega$ assumed to be of class \mathcal{C}^1 . We denote by ν the outward unit normal to Ω on $\Gamma = \partial\Omega$. Let $T > 0$ and $t_0 \in (0, T)$. We shall use the following notations $Q_0 = \Omega \times (0, T)$, $Q = \Omega \times (t_0, T)$, $\Sigma = \Gamma \times (t_0, T)$ and $\Sigma_0 = \Gamma \times (0, T)$. We consider the following heat equation:

$$(1.1) \quad \begin{cases} \partial_t q = \nabla \cdot (c(x) \nabla q) & \text{in } Q_0, \\ q(t, x) = g(t, x) & \text{on } \Sigma_0, \\ q(0, x) = q_0 & \text{in } \Omega. \end{cases}$$

Our problem can be stated as follows:

Inverse Problem

Is it possible to determine the coefficient $c(x)$ from the following measurements:

$$\partial_\nu(\partial_t q)|_{(t_0, T) \times \Gamma_0} \quad \text{and} \quad \nabla(\Delta q(T', \cdot)), \Delta q(T', \cdot), q(T', \cdot) \quad \text{in } \Omega \text{ for } T' = \frac{t_0 + T}{2},$$

where Γ_0 is a part of the boundary Γ of Ω ?

Let q (resp. \tilde{q}) be solution of (1.1) associated to (c, g, q_0) (resp. (\tilde{c}, g, q_0)), we assume

Assumption 1.1. • q_0 belongs to $H^4(\Omega)$ and g is sufficiently regular (e.g. $\exists \epsilon > 0$ such that $g \in H^1(0, T, H^{3/2+\epsilon}(\partial\Omega)) \cap H^2(0, T, H^{5/2+\epsilon}(\partial\Omega))$)

- $c, \tilde{c} \in \mathcal{C}^3(\Omega)$,
- There exist a constant $r > 0$, such that $q_0 \geq r$ and $g \geq r$.

Note that the first item of the previous assumptions implies that (1.1) admits a solution in $H^1(t_0, T, H^2(\Omega))$ (see Lions [12]). We will later use this regularity result. The two last items allows us to state that the function u satisfies $|\Delta q(x, T')| \geq r > 0$ and $|\nabla q(x, T')| \geq r > 0$ in Ω (see Pazy [15], Benabdallah, Gaitan and Le Rousseau [4]).

We assume that we can measure both the normal flux $\partial_\nu(\partial_t q)$ on $\Gamma_0 \subset \partial\Omega$ in the time interval (t_0, T) for some $t_0 \in (0, T)$ and $\nabla(\Delta q)$, Δq and ∇q at time $T' \in (t_0, T)$.

Our main result is a stability result for the coefficient $c(x)$:

For q_0 in $H^2(\Omega)$ there exists a constant $C = C(\Omega, \Gamma, t_0, T, r) > 0$ such that

$$\begin{aligned} |c - \tilde{c}|_{H_0^1(\Omega)}^2 &\leq C |\partial_\nu(\partial_t q) - \partial_\nu(\partial_t \tilde{q})|_{L^2((t_0, T) \times \Gamma_0)}^2 \\ &+ C |\nabla(\Delta q(T', \cdot)) - \nabla(\Delta \tilde{q}(T', \cdot))|_{L^2(\Omega)}^2 \\ &+ C |\Delta q(T', \cdot) - \Delta \tilde{q}(T', \cdot)|_{L^2(\Omega)}^2 + C |\nabla q(T', \cdot) - \nabla \tilde{q}(T', \cdot)|_{L^2(\Omega)}^2. \end{aligned}$$

The key ingredients to this stability result are a global Carleman-type estimate, a Poincaré-type estimate and an energy estimate. We use the classical Carleman estimate with one observation on the boundary for the heat equation obtained in Fernandez-Cara and Guerrero [8], Fursikov and Imanuvilov [9]. Following the method developed by Imanuvilov, Isakov and Yamamoto for the Lamé system in Imanuvilov, Isakov and Yamamoto [11], we give a Poincaré-type estimate. Then, we prove an energy estimate. Such energy estimate has been proved in Lasiecka, Triggiani and Zhang [13] for the Schrödinger operator in a bounded domain in order to obtain a controllability result and in Cristofol, Cardoulis and Gaitan [6] for the Schrödinger operator in an unbounded domain in order to obtain a stability result. Then using these estimates, we give a stability and uniqueness result for the diffusion coefficient $c(x)$. In the perspective of numerical reconstruction, such problems are ill-posed and stability results are thus of importance.

In the stationary case, the inverse conductivity problem has been studied by several authors. There are different approaches. For the two dimensional case, Nachman [14] proved an uniqueness result for the diffusion coefficient $c \in C^2(\overline{\Omega})$ and Astala and Päivärinta [1] for $c \in L^\infty(\Omega)$ with many measurements from the whole boundary. In the three dimensional case, with the use of complex exponentially solutions, Faddeev [7], Calderon [5], Sylvester and Uhlmann [16] showed uniqueness for the diffusion coefficient.

There are few results on Lipschitz stability for parabolic equations, we can cite Imanuvilov and Yamamoto [10], Benabdallah, Gaitan and Le Rousseau [4]. In [4], the authors prove a Lipschitz stability result for the determination of a piecewise-constant diffusion coefficient. For smooth coefficients in the principal part of a parabolic equation, Yuan and Yamamoto [17] give a Lipschitz stability result with multiple observations. This paper is an improvement of the simple case in [17] where we consider that the diffusion coefficient is a real valued function and not a $n \times n$ -matrix. Indeed, in this case, with the method developped by [17], they need two observations in order to obtain an estimation of the H^1 -norm of the diffusion coefficient. In this case, we need only one observation.

Our paper is organized as follows. In Section 2, we recall the global Carleman estimate for (1.1) with one observation on the boundary. Then we prove a Poincaré-type estimate for the coefficient $c(x)$ and an energy estimate. In Section 3, using the previous results, we establish a stability estimate for the coefficient $c(x)$ when one of the solutions \tilde{q} is in a particular class of solutions with some regularity and "positivity" properties.

2 Some Usefull Estimates

2.1 Global Carleman Estimate

We recall here a Carleman-type estimate with a single observation acting on a part Γ_0 of the boundary Γ of Ω in the right-hand side of the estimate (see [8]), [9]. Let us introduce the following notations:

let $\tilde{\beta}$ be a $\mathcal{C}^4(\overline{\Omega})$ positive function such that there exists a positive constant C_0 which satisfies

Assumption 2.1. $|\nabla \tilde{\beta}| \geq C_0 > 0$ in Ω , $\partial_\nu \tilde{\beta} \leq 0$ on $\Gamma \setminus \Gamma_0$,

Then, we define $\beta = \tilde{\beta} + K$ with $K = m\|\tilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (t_0, T)$, we define the weight functions

$$\varphi(x, t) = \frac{e^{\lambda\beta(x)}}{(t - t_0)(T - t)}, \quad \eta(x, t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(t - t_0)(T - t)}.$$

If we set $\psi = e^{-s\eta}q$, we also introduce the following operators

$$\begin{aligned} M_1\psi &= \nabla \cdot (c\nabla\psi) + s^2\lambda^2 c |\nabla\beta|^2 \varphi^2 \psi + s(\partial_t \eta)\psi, \\ M_2\psi &= \partial_t \psi - 2s\lambda\varphi c \nabla\beta \cdot \nabla\psi - 2s\lambda^2 \varphi c |\nabla\beta|^2 \psi. \end{aligned}$$

Then the following result holds (see [8], [9])

Theorem 2.2. *There exist $\lambda_0 = \lambda_0(\Omega, \Gamma_0) \geq 1$, $s_0 = s_0(\lambda_0, T) > 1$ and a positive constant $C = C(\Omega, \Gamma_0, T)$ such that, for any $\lambda \geq \lambda_0$ and any $s \geq s_0$, the following inequality holds:*

$$\begin{aligned} (2.2) \quad & \|M_1(e^{-s\eta}q)\|_{L^2(Q)}^2 + \|M_2(e^{-s\eta}q)\|_{L^2(Q)}^2 \\ & + s\lambda^2 \iint_Q e^{-2s\eta} \varphi |\nabla q|^2 dx dt + s^3 \lambda^4 \iint_Q e^{-2s\eta} \varphi^3 |q|^2 dx dt \\ & \leq C \left[s\lambda \iint_{t_0}^T \int_{\Gamma_0} e^{-2s\eta} \varphi |\partial_\nu q|^2 dx dt + \iint_Q e^{-2s\eta} |\partial_t q - \nabla \cdot (c\nabla q)|^2 dx dt \right], \end{aligned}$$

for all $q \in H^1(t_0, T, H^2(\overline{\Omega}))$ with $q = 0$ on Σ .

2.2 Poincaré-type estimate

We consider the solutions q and \tilde{q} to the following systems

$$(2.3) \quad \begin{cases} \partial_t q = \nabla \cdot (c(x)\nabla q) & \text{in } Q_0, \\ q(t, x) = g(t, x) & \text{on } \Sigma_0, \\ q(0, x) = q_0 & \text{in } \Omega, \end{cases}$$

and

$$(2.4) \quad \begin{cases} \partial_t \tilde{q} = \nabla \cdot (\tilde{c}(x)\nabla \tilde{q}) & \text{in } Q_0, \\ \tilde{q}(t, x) = g(t, x) & \text{on } \Sigma_0, \\ \tilde{q}(0, x) = q_0 & \text{in } \Omega. \end{cases}$$

We set $u = q - \tilde{q}$, $y = \partial_t u$ and $\gamma = c - \tilde{c}$. Then y is solution to the following problem

$$(2.5) \quad \begin{cases} \partial_t y = \nabla \cdot (c(x) \nabla y) + \nabla \cdot (\gamma(x) \nabla (\partial_t \tilde{q})) & \text{in } Q_0, \\ y(t, x) = 0 & \text{on } \Sigma_0, \\ y(0, x) = \nabla \cdot (\gamma(x) \nabla (q_0(x))), & \text{in } \Omega. \end{cases}$$

Note that with (2.3) and (2.4) we can determine $y(T', x)$ and we obtain

$$(2.6) \quad y(T', x) = \nabla \cdot (\gamma(x) \nabla (\tilde{q}(T', x))) + \nabla \cdot (c(x) \nabla (u(T', x))).$$

We use a lemma proved in [11] for Lamé system in bounded domains:

Lemma 2.3. *We consider the first order partial differential operator*

$$P_0 g := \nabla q_0 \cdot \nabla g$$

where q_0 satisfies

$$|\nabla \beta \cdot \nabla q_0| \neq 0.$$

Then there exists positive constant, $s_1 > 0$ and $C = C(\lambda, T')$ such that for all $s \geq s_1$

$$s^2 \lambda^2 \int_{\Omega} e^{-2s\eta(T')} \varphi(T') |g|^2 dx dy \leq C \int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') |P_0 g|^2 dx dy$$

with $T' = \frac{t_0 + T}{2}$, $\eta(T') := \eta(x, T')$, $\varphi(T') := \varphi(x, T')$ and for $g \in H_0^1(\Omega)$.

We assume

Assumption 2.4. $|\nabla \beta \cdot \nabla \tilde{q}(T')| \neq 0$,

Proposition 2.5. *Let \tilde{q} be solution of (2.4). We assume that Assumption 2.4 are satisfied. Then there exists a positive constant $C = C(T', \lambda)$ such that for s large enough ($s \geq s_1$), the following estimate hold true*

$$\begin{aligned} s^2 \lambda^2 \int_{\Omega} e^{-2s\eta(T')} \varphi(T') (|\nabla \gamma|^2 + |\gamma|^2) dx &\leq C \int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') (|\nabla y(T')|^2 + |y(T')|^2) dx \\ &+ C \int_{\Omega} e^{-2s\eta(T')} (|\nabla(\Delta u(T'))|^2 + |\Delta u(T')|^2 + |\nabla u(T')|^2) dx \end{aligned}$$

for $\gamma \in H_0^2(\Omega)$.

Proof. We are dealing with the following first order partial differential operators given by the equation (2.6)

$$P_0(\gamma) := \sum_{i=1}^n \partial_{x_i} \tilde{q}(T') \partial_{x_i} \gamma = y(T') - \gamma \Delta \tilde{q}(T') - \nabla(c \nabla u)(T').$$

We apply the lemma 2.3 for this operator and we can write :

$$\begin{aligned}
(2.7) \quad & s^2 \lambda^2 \int_{\Omega} e^{-2s\eta(T')} \varphi(T') |\gamma|^2 \, dx \leq C \int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') (|y(T')|^2 + |\gamma|^2) \, dx \\
& + C \int_{\Omega} e^{-2s\eta(T')} (|\Delta u(T')|^2 + |\nabla u(T')|^2) \, dx
\end{aligned}$$

In the other hand, we use the x_j -derivative of the previous equation (2.6). So, for each j we deal with the following first order partial differential operator :

$$P_0(\partial_{x_j} \gamma) = \partial_{x_j}(T') - \partial_{x_j} \gamma \Delta \tilde{q}(T') - \gamma \Delta(\partial_{x_j} \tilde{q})(T') - \partial_{x_j}(\nabla(c \nabla u))(T').$$

Then under assumption (2.4):

$$\begin{aligned}
s^2 \lambda^2 \int_{\Omega} e^{-2s\eta(T')} \varphi(T') |\partial_{x_j} \gamma|^2 \, dx & \leq C \int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') |\partial_{x_j} y(T')|^2 \, dx \\
& + C \int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') (|\partial_{x_j} \gamma|^2 + |\gamma|^2 + |\nabla \gamma|^2 + |\partial_{x_j} F|^2) \, dx
\end{aligned}$$

So, adding for all j , we can write

$$\begin{aligned}
(2.8) \quad & s^2 \lambda^2 \int_{\Omega} e^{-2s\eta(T')} \varphi(T') |\nabla \gamma|^2 \, dx \leq C \int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') |\nabla y(T')|^2 \, dx \\
& + C \int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') (|\nabla \gamma|^2 + |\gamma|^2 + |\nabla(\Delta u(T'))|^2 + |\Delta u(T')|^2) \, dx
\end{aligned}$$

Taking into account (2.7) and (2.8) and for s large enough, we can conclude. □

2.3 Estimation of $\int_{\Omega} e^{-2s\eta(T')} |y(T')|^2 \, dx$

Let $T' = \frac{1}{2}(T + t_0)$ the point for which $\Phi(t) = \frac{1}{(t-t_0)(T-t)}$ has its minimum value.

We set $\psi = e^{-s\eta} y$. With the operator

$$(2.9) \quad M_2 \psi = \partial_t \psi - 2s\lambda\varphi c \nabla \beta \cdot \nabla \psi - 2s\lambda^2 \varphi c |\nabla \beta|^2 \psi,$$

we introduce, following [2],

$$\mathcal{I} = \int_{t_0}^{T'} \int_{\Omega} M_2 \psi \, \psi \, dx dt$$

We have the following estimates.

Lemma 2.6. *Let $\lambda \geq \lambda_1$, $s \geq s_1$ and let $a, b, c, d \in L^\infty(\Omega)$. Furthermore, we assume that u_0, v_0 in $H^2(\Omega)$ and the assumption (1.1) is satisfied. Then there exists a constant $C = C(\Omega, \omega, T)$ such that*

$$(2.10) \quad \int_{\Omega} e^{-2s\eta(T', x)} |y(T', x)|^2 dx \leq C \left[\lambda^{1/2} \int_{t_0}^T \int_{\Gamma_0} e^{-2s\eta} \varphi |\partial_\nu y|^2 dx dt \right. \\ \left. + s^{-1/2} \lambda^{-1/2} \int_{t_0}^T \int_{\Omega} e^{-2s\eta} (|\gamma|^2 + |\nabla \gamma|^2) dx dt \right].$$

Proof. If we compute \mathcal{I} , we obtain :

$$\int_{\Omega} e^{-2s\eta(T', x)} |y(T', x)|^2 dx = -2\mathcal{I} \\ -4s\lambda \int_{t_0}^{T'} \int_{\Omega} \varphi c \nabla \beta \cdot \nabla \psi \psi dx dt - 4s\lambda^2 \int_{t_0}^{T'} \int_{\Omega} \varphi c |\nabla \beta|^2 |\psi|^2 dx dt.$$

Then with the Carleman estimate (2.2), we can estimate all the terms in the right hand side of the previous equality and we have

$$\int_{\Omega} e^{-2s\eta(T', x)} |y(T', x)|^2 dx \leq C s^{-3/2} \lambda^{-2} \left(\|M_2 \psi\|^2 + s^3 \lambda^4 \iint_Q e^{-2s\eta} \varphi^3 |y|^2 dx dt \right) \\ + C s^{-1} \lambda^{-1/2} \left(s\lambda \iint_Q e^{-2s\eta} \varphi |\nabla y|^2 dx dt + s^3 \lambda^4 \iint_Q e^{-2s\eta} \varphi^3 |y|^2 dx dt \right) \\ + C s^{-2} \lambda^{-2} \left(s^3 \lambda^4 \iint_Q e^{-2s\eta} \varphi^3 |y|^2 dx dt \right).$$

Finally, we obtain

$$\int_{\Omega} e^{-2s\eta(T', x)} |y(T', x)|^2 dx \leq C \lambda^{1/2} \int_{t_0}^T \int_{\Gamma_0} e^{-2s\eta} \varphi |\partial_\nu y|^2 d\sigma dt + C s^{-1} \lambda^{-1/2} \iint_Q e^{-2s\eta} |f|^2 dx dt,$$

where $f = \nabla \cdot (\gamma \nabla \partial_t \tilde{q})$. We assume that \tilde{q} is sufficiently smooth in order to have $\nabla \partial_t \tilde{q}$ and $\Delta \partial_t \tilde{q}$ in $L^2(O, T, L^\infty(\Omega))$.

Moreover taking into account that $e^{-2s\eta(t)} \leq e^{-2s\eta(T')}$, the proof of Lemma 2.6 is complete. \square

2.4 Estimation of $\int_{\Omega} e^{-2s\eta(T')} \varphi^{-1}(T') |\nabla y(T')|^2 dx$

We introduce

$$(2.11) \quad E(t) = \int_{\Omega} c \varphi^{-1}(x, t) e^{-2s\eta(x, t)} |\nabla y(x, t)|^2 dx.$$

In this section, we give an estimation for the energy $E(t)$ at T' .

Theorem 2.7. *We assume that Assumptions 1.1 are checked, then there exist $\lambda_1 = \lambda_1(\Omega, \omega) \geq 1$, $s_1 = s_1(\lambda_1, T) > 1$ and a positive constant $C = C(\Omega, \Gamma_0, C_0, r, T)$ such that, for any $\lambda \geq \lambda_1$ and any $s \geq s_1$, the following inequality holds:*

$$(2.12) \quad E(T') \leq C \left[s\lambda \int_{t_0}^T \int_{\Gamma_0} e^{-2s\eta} \varphi |\partial_\nu y|^2 dx dt + s \iint_Q e^{-2s\eta} (|\gamma|^2 + |\nabla \gamma|^2) dx dt \right],$$

Proof. We note $f = \nabla \cdot (\gamma(x) \nabla \partial_t \tilde{q})$.

We multiply the first equation of (2.5) by $e^{-2s\eta} \nabla \cdot (c \nabla y) \varphi^{-1}$ and integrate over $(t_0, T) \times \Omega$, we have :

$$(2.13) \quad \int_{t_0}^{T'} \int_{\Omega} \varphi^{-1} e^{-2s\eta} \nabla \cdot (c \nabla y) \partial_t y \, dx \, dt = \int_{t_0}^{T'} \int_{\Omega} \varphi^{-1} e^{-2s\eta} |\nabla \cdot (c \nabla y)|^2 \, dx \, dt \\ + \int_{t_0}^{T'} \int_{\Omega} e^{-2s\eta} \varphi^{-1} \nabla \cdot (c \nabla y) f \, dx \, dt.$$

we denote $A := \int_{t_0}^{T'} \int_{\Omega} e^{-2s\eta} \varphi^{-1} \nabla \cdot (c \nabla y) \partial_t y \, dx \, dt$.

Integrating by parts A with respect to the space variable, we obtain

$$(2.14) \quad A = \int_{t_0}^{T'} \int_{\Omega} c \, e^{-2s\eta} \varphi^{-1} \nabla y \partial_t (\nabla y) \, dx \, dt + 2s\lambda \int_{t_0}^{T'} \int_{\Omega} c \, e^{-2s\eta} \nabla y \partial_t y \nabla \beta \, dx \, dt \\ - \lambda \int_{t_0}^{T'} \int_{\Omega} c \, e^{-2s\eta} \varphi^{-1} \nabla y \partial_t y \nabla \beta \, dx \, dt.$$

Observe that

$$e^{-s\eta} \varphi^{-\frac{1}{2}} \partial_t (\nabla y) = \partial_t (e^{-s\eta} \varphi^{-\frac{1}{2}} \nabla y) + s e^{-s\eta} \varphi^{-\frac{1}{2}} \partial_t \eta \nabla y + \frac{1}{2} e^{-s\eta} \partial_t \varphi \varphi^{-\frac{3}{2}} \nabla y.$$

Hence, the first integral of the right-hand side of (2.14) can be written as

$$(2.15) \quad \int_{t_0}^{T'} \int_{\Omega} c \, e^{-2s\eta} \varphi^{-1} \nabla y \partial_t (\nabla y) \, dx \, dt = \int_{t_0}^{T'} \int_{\Omega} c \, e^{-s\eta} \varphi^{-\frac{1}{2}} \nabla y \partial_t (\nabla y) e^{-s\eta} \varphi^{-\frac{1}{2}} \, dx \, dt \\ = \int_{t_0}^{T'} \int_{\Omega} c \, e^{-s\eta} \varphi^{-\frac{1}{2}} \nabla y \partial_t (e^{-s\eta} \varphi^{-\frac{1}{2}} \nabla y) \, dx \, dt + s \int_{t_0}^{T'} \int_{\Omega} c \, e^{-2s\eta} \varphi^{-1} |\nabla y|^2 \partial_t \eta \, dx \, dt \\ + \frac{1}{2} \int_{t_0}^{T'} \int_{\Omega} c \, e^{-2s\eta} \varphi^{-2} |\nabla y|^2 \partial_t \varphi \, dx \, dt.$$

Using an integration by parts with respect the time variable, the first term of (2.15) is exactly equal to $\frac{1}{2} E(T')$, since $E(t_0) = 0$. Therefore, the equations (2.13), (2.14) and (2.15) yield

$$(2.16) \quad E(T') = -2s \int_{t_0}^{T'} \int_{\Omega} c \, e^{-2s\eta} \varphi^{-1} |\nabla y|^2 \partial_t \eta \, dx \, dt - \int_{t_0}^{T'} \int_{\Omega} c \, e^{-2s\eta} \varphi^{-2} |\nabla y|^2 \partial_t \varphi \, dx \, dt \\ - 4s\lambda \int_{t_0}^{T'} \int_{\Omega} c \, e^{-2s\eta} \nabla y \partial_t y \nabla \beta \, dx \, dt + 2\lambda \int_{t_0}^{T'} \int_{\Omega} c \, e^{-2s\eta} \varphi^{-1} \nabla y \partial_t y \nabla \beta \, dx \, dt \\ + 2 \int_{t_0}^{T'} \int_{\Omega} \varphi^{-1} e^{-2s\eta} |\nabla \cdot (c \nabla y)|^2 \, dx \, dt + 2 \int_{t_0}^{T'} \int_{\Omega} e^{-2s\eta} \varphi^{-1} \nabla \cdot (c \nabla y) f \, dx \, dt \\ = I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

Now, in order to obtain an estimation to $E(T')$, we must estimate all the integrals $I_i, 1 \leq i \leq 6$. Using the fact that $|\partial_t \eta| \leq C(\Omega, \omega)T\varphi^2$, we obtain, in first step, for the integral I_1 , the following estimation

$$\begin{aligned} |I_1| &\leq Cs \int_{t_0}^{T'} \int_{\Omega} c e^{-2s\eta} \varphi |\nabla y|^2 dx dt \\ &\leq C\lambda^{-2} \left[s\lambda^2 \iint_Q e^{-2s\eta} \varphi |\nabla y|^2 dx dt \right]. \end{aligned}$$

In a second step, the Carleman estimate yields

$$|I_1| \leq C\lambda^{-2} \left[s\lambda \int_{t_0}^T \int_{\Gamma_0} e^{-2s\eta} \varphi |\partial_\nu y|^2 dx dt + \iint_Q e^{-2s\eta} |f|^2 dx dt \right],$$

where C is a generic constant which depends on $\Omega, \Gamma_0, c_{\max}$ and T .

As the same way, we have, for I_2 , the following estimate

$$|I_2| \leq Cs^{-1}\lambda^{-2} \left[s\lambda \int_{t_0}^T \int_{\Gamma_0} e^{-2s\eta} \varphi |\partial_\nu y|^2 dx dt + \iint_Q e^{-2s\eta} |f|^2 dx dt \right].$$

The last inequality holds throught the Carleman estimate and the following inequality

$$|\partial_t \varphi| \leq C(\Omega, \Gamma_0)T^3 \frac{\varphi^3}{4}.$$

Using Young inequality, we estimate I_3 .

We have

$$\begin{aligned} |I_3| &\leq Cs \left[s\lambda^2 \iint_Q e^{-2s\eta} \varphi |\nabla y|^2 dx dt + s^{-1} \iint_Q e^{-2s\eta} \varphi^{-1} |\partial_t y|^2 dx dt \right] \\ &\leq Cs \left[s\lambda \int_{t_0}^T \int_{\Gamma_0} e^{-2s\eta} \varphi |\partial_\nu y|^2 dx dt + \iint_Q e^{-2s\eta} |f|^2 dx dt \right], \end{aligned}$$

For the integral I_4 , we have

$$\begin{aligned} |I_4| &\leq C \left[s\lambda^2 \iint_Q e^{-2s\eta} \varphi^{-1} |\nabla y|^2 dx dt + s^{-1} \iint_Q e^{-2s\eta} \varphi^{-1} |\partial_t y|^2 dx dt \right] \\ &\leq C \left[s\lambda \int_{t_0}^T \int_{\Gamma_0} e^{-2s\eta} \varphi |\partial_\nu y|^2 dx dt + \iint_Q e^{-2s\eta} |f|^2 dx dt \right], \end{aligned}$$

where we have used, for the term containing $|\nabla y|^2$, the following estimate

$$\varphi^{-1} \leq C(\Omega, \omega)T^4 \frac{\varphi}{16}.$$

we have immediatly the following estimate for I_5

$$|I_5| \leq Cs \left[s\lambda \int_{t_0}^T \int_{\Gamma_0} e^{-2s\eta} \varphi |\partial_\nu y|^2 dx dt + \iint_Q e^{-2s\eta} |f|^2 dx dt \right].$$

Finally, for the last term I_6 , we have

$$\begin{aligned} |I_6| &\leq C \left[s^{-1} \iint_Q e^{-2s\eta} \varphi^{-2} |\nabla \cdot (c \nabla y)|^2 dx dt + s \iint_Q e^{-2s\eta} |f|^2 dx dt \right] \\ &\leq C \left[s\lambda \int_{t_0}^T \int_{\Gamma_0} e^{-2s\eta} \varphi |\partial_\nu y|^2 dx dt + s \iint_Q e^{-2s\eta} |f|^2 dx dt \right]. \end{aligned}$$

The last inequality holds using the following estimate

$$\varphi^{-2} \leq C(\Omega, \omega) T^2 \frac{\varphi^{-1}}{4}$$

If we come back to (2.16), using the estimations of I_i , $1 \leq i \leq 6$ and expanding the term f , this conclude the proof of Theorem 2.7. \square

3 Stability Result

Theorem 3.1. *Let q and \tilde{q} be solutions of (2.3) and (2.4) such that $c - \tilde{c} \in H_0^2(\Omega)$. We assume that Assumptions 1.1 are satisfied. Then there exists a positive constant $C = C(\Omega, \Gamma_0, T)$ such that for s and λ large enough,*

$$\begin{aligned} \int_\Omega \varphi(T') e^{-2s\eta(T')} (|c - \tilde{c}|^2 + |\nabla(c - \tilde{c})|^2) dx dy &\leq C \int_0^T \int_{\Gamma_0} \varphi e^{-2s\eta} \partial_\nu \beta |\partial_\nu(\partial_t q - \partial_t \tilde{q})|^2 d\sigma dt \\ &\quad + C \int_\Omega e^{-2s\eta(T')} (|\nabla(\Delta u(T'))|^2 + |\Delta u(T')|^2 + \nabla u(T')^2) dx \end{aligned}$$

Proof. Using the estimates (2.12), (2.10) and Proposition (2.5), we obtain

$$\begin{aligned} s^2 \lambda^2 \int_\Omega e^{-2s\eta(T')} \varphi(T') (|\nabla \gamma|^2 + |\gamma|^2) dx &\leq C \int_\Omega e^{-2s\eta(T')} \varphi^{-1}(T') (|\nabla y(T')|^2 + |y(T')|^2) dx \\ &\quad + C \int_\Omega e^{-2s\eta(T')} (|\nabla(\Delta u(T'))|^2 + |\Delta u(T')|^2 + \nabla u(T')^2) dx \\ &\leq C \left[s\lambda \int_{t_0}^T \int_{\Gamma_0} e^{-2s\eta} \varphi |\partial_\nu y|^2 dx dt + s \iint_Q e^{-2s\eta} (|\gamma|^2 + |\nabla \gamma|^2) dx dt \right] \\ &\quad + C \left[\lambda^{1/2} \int_{t_0}^T \int_{\Gamma_0} e^{-2s\eta} \varphi |\partial_\nu y|^2 dx dt + s^{-1/2} \lambda^{-1/2} \int_{t_0}^T \int_\Omega e^{-2s\eta} (|\gamma|^2 + |\nabla \gamma|^2) dx dt \right] \\ &\quad + C \int_\Omega e^{-2s\eta(T')} (|\nabla(\Delta u(T'))|^2 + |\Delta u(T')|^2 + \nabla u(T')^2) dx. \end{aligned}$$

So we get for s sufficiently large

$$\begin{aligned} s^2 \lambda^2 \int_\Omega e^{-2s\eta(T')} \varphi(T') (|\nabla \gamma|^2 + |\gamma|^2) dx &\leq C s \lambda \int_{t_0}^T \int_{\Gamma_0} e^{-2s\eta} \varphi |\partial_\nu y|^2 dx dt \\ &\quad + C \int_\Omega e^{-2s\eta(T')} (|\nabla(\Delta u(T'))|^2 + |\Delta u(T')|^2 + \nabla u(T')^2) dx, \end{aligned}$$

and the the theorem is proved. \square

Remark

- All the previous results are available for $\Omega \subset \mathbb{R}^n$ be a bounded domain of \mathbb{R}^n with $n \geq 3$ if we adapt the regularity properties of the initial and boundary data.
- We give a stability result for two linked coefficient (c and ∇c) with one observation. Note that for two independent coefficients, there is no result in the litterature with only one observation.

References

- [1] K. Astala and L. Päiväranta, *Calderon's inverse conductivity problem in the plane*, Ann. Math., **163**, (2006), 265–299.
- [2] L. Baudouin and J.P. Puel, *Uniqueness and stability in an inverse problem for the Schrödinger equation*, Inverse Problems, **18**, (2002), 1537–1554.
- [3] H. Brezis, "Analyse fonctionnelle", Masson, Paris, 1983.
- [4] A. Benabdallah, P. Gaitan and J. Le Rousseau, *Stability of discontinuous diffusion coefficients and initial conditions in an inverse problem for the heat equation*, Accepted for publication in SIAM J. Control Optim (SICON), 2007.
- [5] A.P. Calderon, *On an inverse boundary value problem*, "Seminar on Numerical Analysis and its Applications to Continuum Physics", Rio de Janeiro,(1980) 65–73.
- [6] L. Cardoulis, M. Cristofol and P. Gaitan, *Inverse problem for the Schrödinger operator in an unbounded strip using a Carleman estimate*, Preprint LATP, 2006, submitted to J. Inverse and Ill-Posed Problems.
- [7] L.D. Faddeev, *Griwing solutions of the Schrödinger equation*, Dokl. Akad. Nauk SSSR, **165**, (1965), 514–517.
- [8] E. Fernández-Cara and S. Guerrero, *Global Carleman estimates for solutions of parabolic systems defined by transposition and some applications to controllability*, Applied Mathematics Research eXpress, **ID 75090**, (2006), 1–31.
- [9] A. Fursikov and O. Yu. Imanuvilov, "Controllability of evolution equations", Seoul National University, Korea, Lecture Notes, 34, 1996.
- [10] O. Yu. Imanuvilov and M. Yamamoto, *Lipschitz stability in inverse problems by Carleman estimates*, Inverse Problems, **14**, (1998), 1229–1245.
- [11] O. Yu. Imanuvilov, V. Isakov and M. Yamamoto, *An Inverse Problem for the Dynamical Lamé system with two set of boundary data*, CPAM, **LVI**, (2003), 1366–1382.
- [12] J.L. Lions, "Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles" Dunod, 1968.
- [13] I. Lasiecka, R. Triggiani and X. Zhang, *Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates*, J. Inv. Ill-Posed Problems, **11**, **3**, (2003), 1–96.
- [14] A. Nachman, *A global uniqueness for a two dimensional inverse boundary problem*, Ann. Math., **142**, (1995), 71–96.

- [15] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations", Springer-Verlag, New York, 1983.
- [16] J. Sylvester and G. Uhlmann, *Global uniqueness theorem for an inverse boundary problem*, Ann. Math., **125**, (1987), 153–169.
- [17] G. Yuan and M. Yamamoto, *Lipshitz stability in the determination of the principal parts of a parabolic equation by boundary measurements*, Preprint Tokyo University, UTMS, **27**, (2006).

E-mail address: gaitan@cmi.univ-mrs.fr